Ch. 4. Properties of Greens's Functions in QM 8/17/09 dsm

4.0 In this chapter we will examine some of the properties of the Green's function, G(E) and its application to topics in QM.

4.1 Continuity of an operator.

The first topic of interest is that of the continuity of the operator, $G(E)$.

An operator, T, is said to be continuous or bounded it, for every vector, f, we have

$$\|Tf\| \leq C\|f\| \qquad (4.1)$$

for some positive constant, C, independent of f. In QM, the free-particle Hamiltonian, H_0 is unbounded, as we have already discussed; which for this discussion means that no such number, C, exists. Furthermore, $H=H_0+V$ is likewise unbounded. Our Green's function, $G(E)$, however, is bounded (almost everywhere), as we shall demonstrate in the next section.

To do this, we require some more definitions:

1. The spectrum of H consists of all values of E for which $G(E)$ is unbounded, or fails to exist. As such, the spectrum contains all the eigenfunctions of H , both continuous and discrete.
2. The resolvent set consists of all values of E, on the first sheet, which are not in the spectrum. That is, all the points where $G(E)$ is bounded.
3. The first sheet is all values of $\sqrt{E}$ for which $Imag\sqrt{E} \geqslant 0$.

4.2 Analyticity of G(E).

In this section we prove that G(E) is analytic in the resolvent set.
We will demonstrate this by showing that all the matrix elements , (f,G(E)g), are analytic.

We need the the identity:

$$-A^{-1}+B^{-1}=A^{-1}(A-B)B^{-1} \qquad (4.2)$$

To demonstrate this, we multiplying through on the RHS:

$$-A^{-1}+B^{-1}=A^{-1}AB^{-1}-A^{-1}BB^{-1}=B^{-1}-A^{-1} \quad \text{QED}$$

Let $G(E_0)=A^{-1}$ $G(E)=B^{-1}$ and substitute:

$$G(E)-G(E_0)=G(E_0)(G(E_0)^{-1}-G(E)^{-1})G(E)$$

Now $G(E_0)^{-1}=H-E_0$ and $G(E)^{-1}=H-E$, thus

$$G(E_0)^{-1}-G(E)^{-1}=H-E_0-(H-E)=E-E_0$$ which combines to yield:

$$G(E)-G(E_0)=G(E_0)(E_0-E)G(E)$$ rearranging, yields the result:

$$G(E) - G(E_0) = G(E_0)G(E)(E_0 - E) \qquad (4.3)$$

Which we can recast, to solve for $G(E)$:

$$G(E) - G(E_0)G(E)(E_0 - E) = G(E_0)$$

and taking into account that $G(E)$ and $G(E_0)$ commute,

$$G(E) - G(E)G(E_0)(E_0 - E) = G(E_0)$$ or:

$$G(E) = G(E_0)(1 - G(E_0)(E_0 - E))^{-1} \qquad (4.4)$$

Now suppose E_0 is not in the spectrum. Them, by definition, $G(E_0)$ is bounded, but let's look at this a little more closely. That is, given:

$$G(E_0) = \frac{1}{H - E_0}$$

And also given that $G(E_0)$ is a linear operator in a Hilbert space; and given that E_0 is not in the spectrum.

Let's look at the operation: $G(E_0)|a\rangle = \frac{1}{H - E_0}|a\rangle$

$$= \frac{1}{E_a - E_0}|a\rangle$$

And because we have defined $E_0 \notin E_a$, then we see that everything looks fine, and that $G(E_0)$ is in fact bounded. And thus we say with confidence that $G(E_0)$ is continuous. That is,

$$\|G(E_0)f\| \leq C\|f\| \qquad (4.5)$$

for all vectors f .

Now we will employ the fact of the continuity of $G(E_0)$ to examine $G(E)$.
Consider the formal power series implied by (4.4) above, for an arbitrary matrix element:

$$(f, G(E)g) = \sum_{n=0}^{\infty} (f, G(E_0)^{n+1} g)(E_0 - E)^n \qquad (4.6)$$

Now we take the norm of the above and apply the Cauchy-Hadamard theorem (2.20),

$$|(f, G(E)g)| = \sum_{n=0}^{\infty} |(f, G(E_0)^{n+1} g)||(E_0 - E)^n| \quad (4.7)$$

And, let's consider the nth term in the above summation.

$$|(f, G(E_0)^{n+1} g)| \leqslant \|f\| \|G(E_0)^{n+1} g\| \quad (4.8)$$

The above inequality follows from the Schwarz inequality (2.19).
We change our nomenclature a little bit here and let $G_0 = G(E_0)$.
And given the boundedness of G_0 , (4.3), we can expand the n = 1 term in (4.6) per the following:

$$\|G_0^2 f\| \le C \|G_0 f\| \le C^2 \|f\| \quad (4.9)$$

This we generalize to:

$$\|G_0^n f\| \le C^n \|f\| \quad (4.10)$$

This is just the convergence criteria we need, as long as:

$$C^n < \frac{1}{(E - E_0)} \quad (4.11)$$

And since C is finite around E_0 (given $E \neq E_0$), then there is a circle of finite size about E_0 in which $(f, G(E)g)$ is analytic.

4.3 The spectrum of H

We can show that $G(E)$ is analytic everywhere in the first sheet except for some segments of the real axis. This amounts to showing that the spectrum of H is only real. To show this, we start with the identity (4.2) with $G(E)^{-1} = B = E - H \quad G^+(E)^{-1} = A = E - H = B^+$.

Substituting into (4.2) gives,

$$-G^+(E) + G(E) = G^+(E)(E^* - H - E + H)G(E)$$

Which reduces to,

$$G(E) - G^+(E) = G^+(E)G(E)(E^* - E) \quad (4.12)$$

Reformulating $(E^* - E)$ into real and imaginary parts,

$$(E^* - E) = E_r - iE_i - E_r + iE_i = i2ImgE \quad (4.13)$$

Substituting (4.13) into (4.12) yields,

$$G(E)-G^{+}(E)=-i2ImgEG(E) \qquad (4.14)$$

Now we find the diagonal matrix element of (4.14), (and neglect showing E dependence),

$$-i2ImgE(Gf, Gf)=(f, Gf)-(Gf, f) \quad (4.15)$$

And since $(f, Gf)=(Gf, f)^{*}$ the RHS of (4.15) reduces to,

$$-i2Img(f, Gf) \quad (4.16)$$

Also, clearly $|Img(f, Gf)|\leq|(f, Gf)|\leq\|f\|\|Gf\|$ (4.17)

So, by the Swartz inequality, we find the absolute value of (4.15),

$$|ImgE|\circ\|Gf\|^{2}\leq\|f\|\circ\|Gf\| \qquad (4.18)$$

And the constant, C, above, can be chosen to be $|ImgE|^{-1}$.
This completes the proof, that the power series, (4.7), converges for every circle not intersecting the real axis. So, $G(E)$ is analytic except on the real axis.